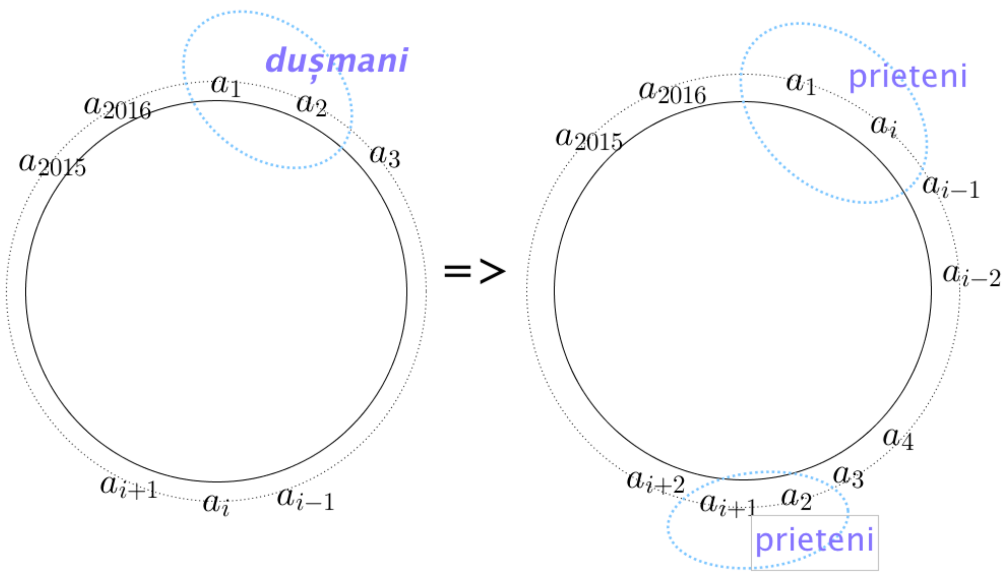
Solutii clasa a VII-a

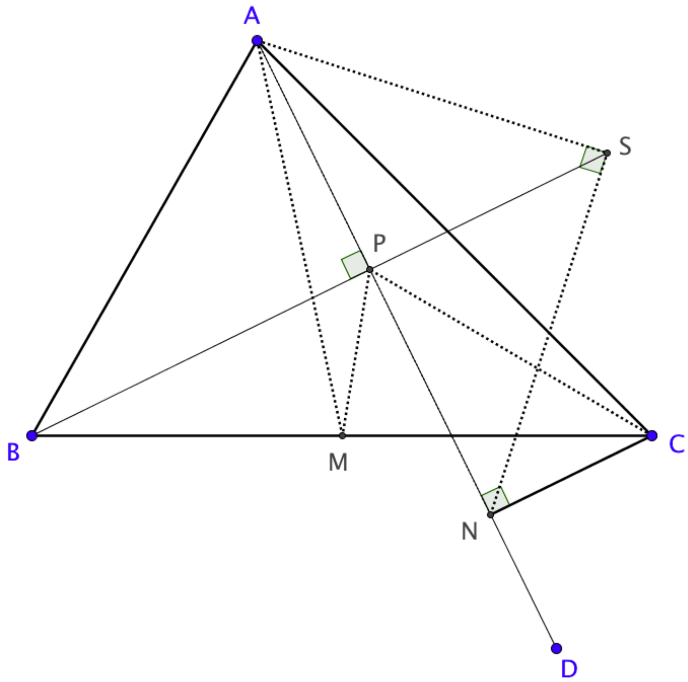
1. Notam x=d·a; y=d·b; (a,b)=1=> $d^{n+1}\cdot a\cdot b^n = p\cdot d(a+b)$ => $d^n\cdot a\cdot b^n=p(a+b)$ => a+b | $d^n\cdot a\cdot b^n$; (a,b)=1=>(a, a+b)=(a+b, b^n)=(a,b)=1=> a+b | d^n => $d^n=(a+b)\cdot k$ => $k\cdot a\cdot b^n=p$
p prim => b=1 => i) a=1
ii)p=k

   i) a=b=1=>x=y=>$x^{n+1}=2\cdot x\cdot p$ => $x^n=2\cdot p$ => p=2; $x^n=4$ fals ($n\geq 3$)
   ii) p=k, b=1 => $d^n\cdot a=p(a+1)$ => a| p(a+1); (a,a+1)=1 => a|p, p prim => a=1 (nu convine) sau a=p => $d^n=p+1$=> $p=d^n-1$ =>
   $p=(d-1)(d^{n-1}+d^{n-2}+...+d+1)$ prim => d=2 => $p=2^n-1$; x=2·p; y=2;

   a) n=3 => p=7, x=14, y=2;
   n=5 => p=31, x=62, y=2;
   b) Pp.ca n este compus n=u·v => $p=2^{u\cdot v}-1=(2^u)^v-1=(2^u-1)((2^u)^{v-1}+...+2^4+1)$.
   contradictie.

2. Fie ($a_1,a_2,...,a_{2016},a_1$) o asezare astfel incat a_1 si a_2 sa fie dusmani. Vom demonstra ca putem sa facem o rearanjare astfel incat numarul de perechi de dusmani sa scada fara a afecta celelalte perechi.
   a_1 va avea printre diplomatii $a_3,a_4,a_5,...,a_{2015}$ cel putin 1008 prieteni.
   Vom demonsta ca exista printre cei 1008 prieteni ai lui a_1 un vecin la dreapta care este prieten cu a_2.
   Dar in dreapta celor 1008 prieteni ai lui a_1 nu pot fi 1008 dusmani ai lui a_2. Fie a_i,a_{i+1}, astfel incat (a_1,a_i) si (a_2,a_{i+1}) sunt prieteni. Vom face urmatoarea rearanjare:





3. Aplicam teorema medianei

=> $AM^2 = \frac{2(AB^2+AC^2)-BC^2}{4}$

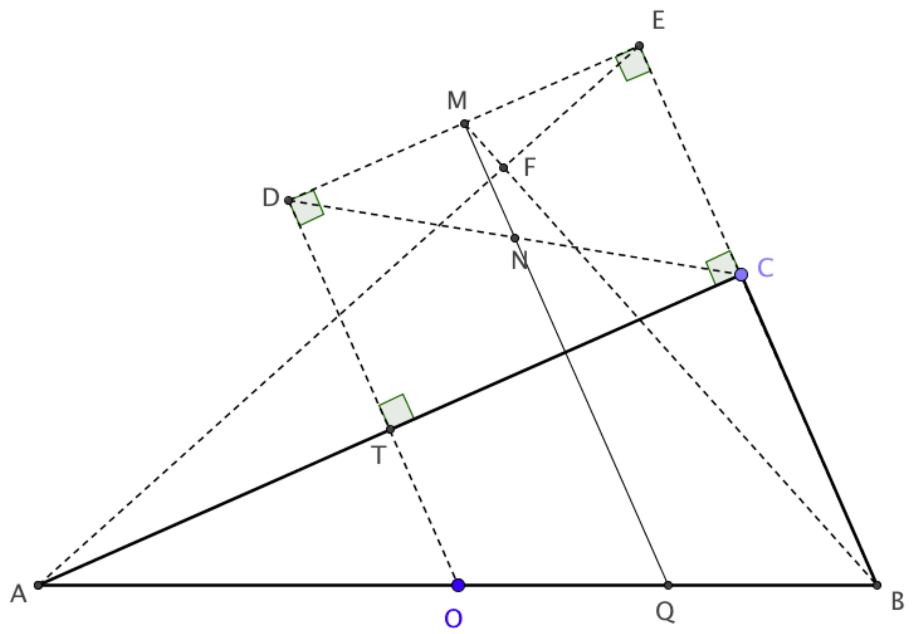
$PM^2 = \frac{2(BP^2+CP^2)-BC^2}{4}$

=> $AM^2 - PM^2 = \frac{2(AB^2+AC^2-BP^2-CP^2}{4} = \frac{AP^2+AC^2-CP^2}{2} = \frac{AP^2+AN^2+NC^2-CP^2}{2}$

$= \frac{AP^2+AN^2-PN^2}{2} = \frac{AP^2+(AP+PN)^2-PN^2}{2} = \frac{2AP^2+2AP\cdot PN}{2}$

$= AP\cdot AN = AS^2$

=> $AM^2 = PM^2 + AS^2$ => putem forma un triunghi dreptunghic.



4. Notam cu O centrul semicercului si OD ∩ AC ={T}. D mijlocul arcului $\widehat{AC}$ => OD ⊥ AC, $\widehat{ACB}$=90˚ ($\measuredangle$ $inscris\ in\ semicerc$); $\widehat{DEC}$=90° => DECT dreptunghi => DE⊥DO => MD tangenta la cerc => MF·MB=MD^2 (puterea punctului M fata de cerc), MF·MB=ME^2 (teorema catetei) => MD^2=ME^2 => MD=ME => MN linie mijlocie in △DEC. Dar ODEB trapez => MQ linie mijlocie in trapez (Q∈[OB]) => MN trece prin mijlocul segmentului OB.